

# DIFFERENTIAL CALCULUS ON A THREE-PARAMETER OSCILLATOR ALGEBRA

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## Abstract

Two differential calculi are developed on an algebra generalizing the usual  $q$ -oscillator algebra and involving three generators and three parameters. They are shown to be invariant under the same quantum group that is extended to a ten-generator Hopf algebra. We discuss the special case where it reduces to a deformation of the invariance group of the Weyl-Heisenberg algebra for which we prove the existence of a constraint between the values of the parameters.

## 1 Introduction

Generalizing the differential geometry on Lie groups and manifolds, the differential calculus on quantum groups and quantum spaces was developed in many interesting papers , (see for example [1],[2],[3],[4]). An exterior differential, one-forms and partial derivatives are defined on a quadratic algebra, that is, a free associative algebra generated by variables satisfying quadratic commutation relations. In the present paper, as in [10] and [11], we consider

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the physically important case where the quadratic algebra is a deformation of the Weyl-Heisenberg algebra, in order to obtain a differential calculus on the algebra of the observables of a quantum system.

The Weyl-Heisenberg algebra can be seen as a free associative algebra generated by three variables  $x^i$  satisfying the quadratic relations :

$$R : \begin{cases} x^1 x^2 - x^2 x^1 - s(x^3)^2 &= 0 \\ x^1 x^3 - x^3 x^1 &= 0 \\ x^2 x^3 - x^3 x^2 &= 0 \end{cases} \quad (1)$$

where  $\hat{a} = x^1$ ,  $\hat{a}^\dagger = x^2$  and  $x^3$  is the identity. This algebra is denoted by  $C < x > / R$ .

This space is invariant under the seven-dimensional Lie subgroup of  $GL(3)$ ,  $G$ , constituted by the matrices  $T$  such as  $t_1^3 = t_2^3 = 0$  and  $t_1^1 t_2^2 - t_2^1 t_1^2 = (t_3^3)^2$ .

The quadratic deformation of the Weyl-Heisenberg algebra considered in this paper, as in [10] and [11], is an algebra, generalizing  $C < x > / R$ , obtained by replacing the relations (1) by

$$R_{xx} : \begin{cases} x^1 x^2 - q x^2 x^1 - s(x^3)^2 &= 0 \\ x^1 x^3 - u x^3 x^1 &= 0 \\ x^2 x^3 - u^{-1} x^3 x^2 &= 0 \end{cases} \quad (2)$$

When  $q = u^{-2}$ , putting  $x^1 = \hat{a}$ ,  $x^2 = \hat{a}^\dagger$  and  $x^3 = q^{-\frac{N}{2}}$ , the algebra  $C < x > / R_{xx}$  is identified to the q-oscillator algebra [8][9]. When  $s = 0$ ,  $C < x > / R_{xx}$  is the three-dimensional quantum plane [5].

In [10] and [11], we establish that a differential calculus on  $C < x > / R_{xx}$  invariant under a seven-dimensional quantum group, deformation of  $G$ , only exists if  $q = u^2$  and then is unique. As the constraint,  $q = u^2$ , eliminates the important case when the quantum space is the q-oscillator algebra, the aim of this paper is to obtain a differential calculus on  $C < x > / R_{xx}$  not restricted, a priori, by a requirement of invariance and valid for arbitrary values of the deformation parameters  $q, u$  and  $s$ .

In section 2, we determine two different sets of consistent quadratic relations between variables, differentials and derivatives. In section 3, we prove that all these relations are invariant under the same quantum group that is extended to a ten-generator Hopf-algebra. When we assume that the variables  $x^1$  and  $x^2$  are mutually adjoint and that  $x^3$  is self-adjoint, the quantum group is endowed with a structure of Hopf-star-algebra. Finally we discuss

the particular case where the quantum group is a deformation of  $G$  and we recover the constraint on the parameters [10], [11]. In this particular case, the ten-generator Hopf-star-algebra contains a eight-generator subalgebra, both algebras leave invariant the commutation relations defining the differential calculus.

## 2 Differential Calculus

Following the usual method [3][4], we add to the free algebra  $C < x >$ , three generators  $\xi^i$ ,  $i = 1, 2, 3$ , identified to the one-forms. We define the exterior differential operator  $d$  in  $C < x, \xi >$  such as :

$$d(x^i) = \xi^i,$$

$$d \text{ is linear,}$$

$$d^2 = 0$$

and  $d$  satisfies the graded Leibniz rule :

$$d(fg) = (df)g + (-1)^k f(dg) \quad (3)$$

where  $f, g \in C < x, \xi >$  and  $f$  is of degree  $k$ . Then, the partial derivatives  $\partial_i$  are defined by :

$$d \equiv \xi^i \partial_i.$$

The commutation relations  $R_{x\xi}$  between the variables and the differentials and those between the partial derivatives  $\partial_k$  and the forms  $\xi^l$  are assumed to be quadratic [3][4] :

$$R_{x\xi} \quad : \quad x^k \xi^l = C_{mn}^{kl} \xi^m x^n \quad (4)$$

and

$$R_{\partial\xi} \quad : \quad \partial_k \xi^l = K_{kn}^{lm} \xi^n \partial_m \quad (5)$$

By applying operator  $d$  to (4) on the left, we get :

$$R_{\xi\xi} \quad : \quad \xi^k \xi^l = -C_{mn}^{kl} \xi^m \xi^n, \quad (6)$$

The above definition implies :  $\partial_l(x^k) = \delta_l^k$ . Applying the Leibniz rule on  $x^k f$ ,  $f \in C < x >$ , and taking into account relations (4), we obtain the commutation relations between  $\partial_i$  and  $x^k$  [3]:

$$R_{x\partial} \quad : \quad \partial_l x^k = \delta_l^k + C_{ln}^{km} x^n \partial_m \quad (7)$$

The combinations of the relations  $R_{\partial\xi}$ ,  $R_{x\partial}$  and  $R_{x\xi}$  lead to

$$K=C^{-1} \quad (8)$$

and to a sufficient condition of consistency, the Yang-Baxter equation :

$$(C \otimes 1)(1 \otimes C)(C \otimes 1) = (1 \otimes C)(C \otimes 1)(1 \otimes C) \quad (9)$$

where 1 is the identity of  $GL(3)$ .

From this point, we simplify the matrix  $C$  assuming that  $C_{lm}^{ij}$  is zero if  $(i, j) \neq (l, m)$  or  $(m, l)$  except  $C_{33}^{12}$  and  $C_{33}^{21}$ . We multiply the relations  $R_{xx}$  on the left by  $\partial_i$ . Using  $R_{x\partial}$ , we commute  $\partial_i$  to the right, and obtain several relations between the elements  $C_{kl}^{ij}$ . In particular,

$$\begin{aligned} C_{12}^{12} &= qC_{12}^{21} - 1, & C_{21}^{12} &= qC_{21}^{21} + q \\ C_{13}^{13} &= uC_{13}^{31} - 1, & C_{31}^{13} &= uC_{31}^{31} + u \\ C_{23}^{32} &= uC_{23}^{23} + u, & C_{32}^{32} &= uC_{32}^{23} - 1 \\ C_{33}^{12} &= qC_{33}^{21} + sC_{33}^{33} + s \end{aligned} \quad (10)$$

and

$$C_{12}^{12}C_{21}^{21} = 0, \quad C_{13}^{13}C_{31}^{31} = 0, \quad C_{23}^{23}C_{32}^{32} = 0 \quad (11)$$

We substitute in the equation (9) a matrix  $C$  satisfying (10). This gives  $27 \times 27$  new relations between the coefficients  $C_{kl}^{ij}$ . Solving these relations together with (11), we find only two different solutions :  $C$  is equal to  $\Omega$  or to its inverse, with  $\Omega$  defined by :

$$\Omega = \begin{pmatrix} q/u^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^2/u^2 & 0 & 0 & 0 & 0 & qs/u^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & q/u & 0 & 0 \\ 0 & q^{-1} & 0 & q/u^2 - 1 & 0 & 0 & 0 & 0 & -s/q \\ 0 & 0 & 0 & 0 & q/u^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q/u^2 - 1 & 0 & 1/u & 0 \\ 0 & 0 & 1/u & 0 & 0 & 0 & q/u^2 - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q/u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q/u^2 \end{pmatrix} \quad (12)$$

The matrices  $\Omega$  and  $\Omega^{-1}$  have the same eigenspaces that correspond to the variable quantum space defined by  $R_{xx}$  and to the one-form quantum space

defined by :

$$R_{\xi\xi} : \begin{cases} (\xi^1)^2 = 0, & (\xi^2)^2 = 0, \\ (\xi^3)^2 = 0, & \xi^2\xi^1 = -u^2/q^2 \xi^1\xi^2 \\ \xi^1\xi^3 = -q/u \xi^3\xi^1, & \xi^2\xi^3 = -u/q \xi^3\xi^2 \end{cases} \quad (13)$$

When  $C$  is equal to  $\Omega$  and to its inverse, the eigenspaces of the transpose matrix  $(C^{-1})^t$  are the same. The six-dimensional eigenspace is identified to the derivative quantum space and is defined by :

$$R_{\partial\partial} : \begin{cases} \partial_1\partial_2 = u^2/q^2 \partial_2\partial_1, \\ \partial_1\partial_3 = u/q \partial_3\partial_1, \\ \partial_2\partial_3 = q/u \partial_3\partial_2 \end{cases} \quad (14)$$

The three-dimensional eigenspace corresponds to the covariant differential forms. We denote  $R$  the set of relations (2), (13) and (14). Corresponding to  $C = \Omega$  or  $C = \Omega^{-1}$ , we obtain two sets of relations  $R_{x\xi}$ ,  $R_{\partial\xi}$  and  $R_{\partial x}$  and then two different differential calculi :

- The set of relations  $R^\Omega$  associated to  $\Omega$  is :

The commutation relations  $R_{x\xi}^\Omega$  between the variables and the differentials :

$$\begin{aligned} x^i\xi^i &= q/u^2 \xi^i x^i, \quad i = 1, 2, 3, & x^1\xi^3 &= q/u \xi^3 x^1, \\ x^1\xi^2 &= q^2/u^2 \xi^2 x^1 + qs/u^2 \xi^3 x^3, & x^3\xi^2 &= q/u \xi^2 x^3, \\ x^2\xi^3 &= (q/u^2 - 1) \xi^2 x^3 + 1/u \xi^3 x^2, & x^3\xi^1 &= (q/u^2 - 1) \xi^3 x^1 + 1/u \xi^1 x^3 \\ x^2\xi^1 &= 1/q \xi^1 x^2 + (q/u^2 - 1) \xi^2 x^1 - s/q \xi^3 x^3, \end{aligned} \quad (15)$$

The commutation relations  $R_{\partial\xi}^\Omega$  between the derivatives and the differentials :

$$\begin{aligned} \partial_3\xi^3 &= (u^2/q - 1) \xi^2\partial_2 + u^2/q \xi^3\partial_2 & \partial_1\xi^2 &= u^2/q^2 \xi^2\partial_1, \\ \partial_1\xi^3 &= u/q \xi^3\partial_1, & \partial_2\xi^1 &= q \xi^1\partial_2, \\ \partial_3\xi^2 &= u/q \xi^2\partial_3 - su^2/q^2 \xi^3\partial_1, & \partial_2\xi^3 &= u \xi^3\partial_2, \\ \partial_3\xi^1 &= u \xi^1\partial_3 + s \xi^3\partial_2, & \partial_2\xi^2 &= u^2/q \xi^2\partial_2, \\ \partial_1\xi^1 &= u^2/q \xi^1\partial_1 + (u^2/q - 1) \xi^3\partial_3 + (u^2/q - 1) \xi^2\partial_2, \end{aligned} \quad (16)$$

The commutation relations  $R_{x\partial}^\Omega$  between the derivatives and the variables :

$$\begin{aligned} \partial_1 x^1 &= 1 + q/u^2 x^1\partial_1, & \partial_2 x^3 &= q/u x^3\partial_2, \\ \partial_3 x^3 &= 1 + q/u^2 x^3\partial_3 + (q/u^2 - 1) x^1\partial_1, & \partial_1 x^2 &= 1/q x^2\partial_1, \\ \partial_3 x^1 &= q/u x^1\partial_3 + qs/u^2 x^3\partial_2, & \partial_2 x^1 &= q^2/u^2 x^1\partial_2, \\ \partial_3 x^2 &= 1/u x^2\partial_3 - s/q x^3\partial_1, & \partial_1 x^3 &= 1/u x^3\partial_1, \\ \partial_2 x^2 &= 1 + q/u^2 x^2\partial_2 + (q/u^2 - 1) & x^1\partial_1 + (q/u^2 - 1) x^3\partial_3. \end{aligned} \quad (17)$$

- The set of relations associated to  $\Omega^{-1}$  :

The commutation relations  $R_{x\xi}^{\Omega^{-1}}$  between the variables and the differentials :

$$\begin{aligned}
x^i \xi^i &= u^2/q \xi^i x^i, \quad i = 1, 2, 3, & x^1 \xi^3 &= (u^2/q - 1) \xi^1 x^3 + u \xi^3 x^1, \\
x^3 \xi^1 &= u/q \xi^1 x^3, & x^2 \xi^1 &= u^2/q^2 \xi^1 x^2 - su^2/q^2 \xi^3 x^3, \\
x^2 \xi^3 &= u/q \xi^3 x^2, & x^3 \xi^2 &= (u^2/q - 1) \xi^3 x^2 + u \xi^2 x^3, \\
x^1 \xi^2 &= (u^2/q - 1) \xi^1 x^2 + q \xi^2 x^1 + s \xi^3 x^3.
\end{aligned} \tag{18}$$

The commutation relations  $R_{\partial\xi}^{\Omega^{-1}}$  between the derivatives and the differentials :

$$\begin{aligned}
\partial_1 \xi^1 &= q/u^2 \xi^1 \partial_1, & \partial_3 \xi^2 &= 1/u \xi^2 \partial_3 - s/q \xi^3 \partial_1, \\
\partial_1 \xi^3 &= 1/u \xi^3 \partial_1, & \partial_2 \xi^1 &= q^2/u^2 \xi^1 \partial_2, \\
\partial_2 \xi^3 &= q/u \xi^3 \partial_2, & \partial_3 \xi^1 &= q/u \xi^1 \partial_3 + sq/u^2 \xi^3 \partial_2, \\
\partial_1 \xi^2 &= 1/q \xi^2 \partial_1, & \partial_3 \xi^3 &= (q/u^2 - 1) \xi^1 \partial_1 + q/u^2 \xi^3 \partial_3, \\
\partial_2 \xi^2 &= (q/u^2 - 1) \xi^1 \partial_1 + (q/u^2 - 1) \xi^3 \partial_3 + q/u^2 \xi^2 \partial_2
\end{aligned} \tag{19}$$

The commutation relations  $R_{x\partial}^{\Omega^{-1}}$  between the derivatives and the variables :

$$\begin{aligned}
\partial_2 x^2 &= 1 + u^2/q x^2 \partial_2, & \partial_1 x^3 &= u/q x^3 \partial_1, \\
\partial_3 x^3 &= 1 + u^2/q x^3 \partial_3 + (u^2/q - 1) x^2 \partial_2, & \partial_2 x^1 &= q x^1 \partial_2, \\
\partial_1 x^2 &= u^2/q^2 x^2 \partial_1, & \partial_3 x^2 &= u/q x^2 \partial_3 - su^2/q x^3 \partial_1, \\
\partial_3 x^1 &= u x^1 \partial_3 + s x^3 \partial_2, & \partial_2 x^3 &= u x^3 \partial_2, \\
\partial_1 x^1 &= 1 + u^2/q x^1 \partial_1 + (u^2/q - 1) x^2 \partial_2 + (u^2/q - 1) x^3 \partial_3.
\end{aligned} \tag{20}$$

The two sets of relations  $R_{xx}, R_{\xi\xi}, R_{\partial\partial}$  with  $R^\Omega$  or  $R^{\Omega^{-1}}$  define two quadratic algebras  $:C < x, \xi, \partial > / R \cup R^\Omega$  and  $C < x, \xi, \partial > / R \cup R^{\Omega^{-1}}$ . In the following section, we investigate their invariance.

It is to be noted that all the construction of the differential calculus is performed without using the B-matrix associated with the variables [3] [4], and is the result solely of the relations  $R_{xx}, R_{x\xi}$  and  $R_{\partial\xi}$ . Moreover, as a consequence of the construction,  $B$  is found to be equal to  $C$ .

### 3 Quantum Group and Invariance

The quantum matrix  $T$  with nine non commuting elements defines a homomorphism on  $C < x, \xi, \partial > [7]$ . The variables  $x$  and the differentials  $\xi$  are transformed by  $T$  and the derivatives  $\partial$  are transformed by  $(T^{-1})^t$ .

When the matrix  $T$  satisfies

$$R_{kl}^{ji} t_m^k t_n^l = t_l^j t_k^i R_{mn}^{lk} \quad (21)$$

with  $R = \Omega$  (resp.  $R = \Omega^{-1}$ ), the relations  $R \cup R^\Omega$  (resp.  $R \cup R^{\Omega^{-1}}$ ) are invariant, and therefore this homomorphism maps  $C < x, \xi, \partial > / R \cup R^\Omega$  (resp.  $C < x, \xi, \partial > / R \cup R^{\Omega^{-1}}$ ) in itself. It is easy to see that  $R = \Omega$  and  $R = \Omega^{-1}$  define the same quantum matrix  $T$ , the elements of which satisfy the following commutation relations  $R_{tt}$  :

$$\begin{aligned} t_2^1 t_1^1 &= q^2 / u^2 t_1^1 t_2^1, & t_2^2 t_1^1 &= t_1^1 t_2^2 - (u^2 - q) / q^2 t_2^1 t_1^2 - qs / u^2 t_1^3 t_2^3, \\ t_3^1 t_2^1 &= u / q t_2^1 t_3^1, & t_3^2 t_1^1 &= 1 / q t_1^1 t_2^2 - s / q (t_1^3)^2, \\ t_3^3 t_1^1 &= q / u t_1^1 t_3^1, & t_3^4 t_1^1 &= u / q t_1^1 t_2^2 - (u^2 - q) / q^2 t_3^1 t_1^2 - s / q t_3^3 t_1^3, \\ t_2^3 t_2^2 &= u t_2^2 t_3^2, & t_3^3 t_1^2 &= t_1^1 t_3^3 - (u^2 - q) / (uq) t_3^1 t_1^3, \\ t_1^3 t_1^1 &= 1 / u t_1^1 t_1^3, & t_3^3 t_1^2 &= q / u t_1^1 t_2^2 - (u^2 - q) / u t_2^1 t_1^3, \\ t_3^3 t_2^1 &= 1 / q t_2^1 t_3^1, & t_2^2 t_2^2 &= 1 / q t_2^2 t_2^2 - s / q (t_2^3)^2, \\ t_3^3 t_2^2 &= u / q t_2^2 t_3^2, & t_3^3 t_2^2 &= u / q^2 t_2^2 t_3^2 - s / q t_3^3 t_2^2, \\ t_1^3 t_2^1 &= u / q^2 t_2^1 t_1^3, & t_1^2 t_2^1 &= u^2 / q^3 t_2^1 t_1^2 - s / q t_1^3 t_2^3, \\ t_2^3 t_2^1 &= q^2 / u t_1^1 t_2^2, & t_3^3 t_2^1 &= t_1^1 t_3^3 - (u^2 - q) / (uq) t_2^1 t_3^3, \\ t_2^3 t_2^2 &= 1 / u t_2^2 t_3^2, & t_3^3 t_2^2 &= 1 / q t_3^2 t_3^2 - s / q (t_3^3)^2 + s / q t_1^1 t_2^2 - su^2 / q^3 t_2^1 t_1^2, \\ t_1^3 t_3^1 &= 1 / q t_3^1 t_1^3, & t_2^2 t_3^1 &= u / q t_3^1 t_2^2 - (u^2 - q) / q^2 t_2^1 t_3^2 - s / u t_3^3 t_2^3, \\ t_1^2 t_2^2 &= u^2 / q^2 t_2^2 t_1^2, & t_3^3 t_2^2 &= t_2^2 t_3^3 + (u^2 - q) / u t_3^2 t_3^2, \\ t_2^3 t_2^1 &= q / u t_2^1 t_3^2, & t_3^3 t_2^2 &= u t_3^2 t_3^3 + sq / u t_1^1 t_2^2 - su t_2^2 t_1^3, \\ t_2^3 t_3^1 &= q^2 / u^2 t_1^1 t_2^2, & t_1^2 t_2^2 &= u / q t_2^2 t_1^2 + (u^2 - q) / u t_1^2 t_2^2, \\ t_2^3 t_3^2 &= q / u t_3^2 t_2^2, & t_1^2 t_3^2 &= t_3^2 t_1^2 + (u^2 - q) / u t_1^2 t_3^2, \\ t_1^3 t_1^2 &= u t_1^2 t_1^3, & t_1^2 t_3^2 &= u / q^2 t_3^2 t_1^2 - su / q^2 t_3^3 t_1^3, \\ t_1^3 t_3^1 &= u / q t_3^1 t_1^3, & t_3^3 t_1^2 &= 1 / u t_3^1 t_3^2 + s / u t_1^1 t_2^2 - su / q^2 t_2^1 t_1^3, \\ t_3^3 t_1^2 &= q t_1^2 t_3^2, & t_2^2 t_3^2 &= q t_3^2 t_2^2 \end{aligned} \quad (22)$$

Any elements of  $C < t > / R_{tt}$  can be written as a sum of ordered monomials  $(t_1^1)^{k_1} (t_2^1)^{k_2} (t_3^1)^{k_3} (t_2^2)^{k_4} (t_1^2)^{k_5} (t_3^2)^{k_6} (t_3^3)^{k_7} (t_1^3)^{k_8} (t_2^3)^{k_9}$  by using the relations  $R_{tt}$ . The inverse  $T^{-1}$ , of  $T$  is equal to :

$$\begin{pmatrix} t_2^2 t_3^3 - q / u t_2^2 t_3^2 & -q^2 / u^2 t_2^1 t_3^3 + q^3 / u^3 t_3^1 t_2^2 & t_2^1 t_3^2 - q / u t_3^1 t_2^2 \\ -u^2 / q^2 t_1^1 t_3^3 + u^3 / q^3 t_3^1 t_1^2 & t_1^1 t_3^3 - u / q t_3^1 t_1^2 & -u^2 / q^2 t_1^1 t_3^2 + u^3 / q^3 t_3^1 t_1^2 \\ t_1^2 t_2^2 - u^2 / q^2 t_2^1 t_1^2 & -q^2 / u^2 t_1^1 t_2^2 + t_2^1 t_1^2 & t_1^1 t_2^2 - u^2 / q^2 t_2^1 t_1^2 \end{pmatrix} D^{-1} \quad (23)$$

with the determinant of  $T$  equal to

$$D = t_1^1 t_2^2 t_3^3 + t_3^1 t_1^2 t_2^3 + u^3 / q^3 t_2^1 t_3^2 t_1^3 - q / u t_1^1 t_3^2 t_2^3 - u^2 / q^2 t_2^1 t_1^2 t_3^3 - u^2 / q^2 t_3^1 t_2^2 t_1^3.$$

We can calculate and verify that  $D$  is not a central element of  $C < t >$  and therefore we have to add the generator  $D^{-1}$  to this algebra. The commutation relations  $R_{tD^{-1}}$  of  $D^{-1}$  with all the  $t_j^i$  are deduced from those of  $D$  :

$$\begin{aligned} t_1^1 D^{-1} &= D^{-1} t_1^1, & t_2^1 D^{-1} &= u^2 / q^4 D^{-1} t_2^1, & t_3^1 D^{-1} &= u / q^2 D^{-1} t_3^1, \\ t_2^2 D^{-1} &= D^{-1} t_2^2, & t_1^2 D^{-1} &= q^2 D^{-1} t_1^2, & t_3^2 D^{-1} &= u / q^2 D^{-1} t_3^2, \\ t_1^3 D^{-1} &= q^2 / u D^{-1} t_1^3, & t_2^3 D^{-1} &= u / q^2 D^{-1} t_2^3, & t_3^3 D^{-1} &= D^{-1} t_3^3. \end{aligned} \quad (24)$$

The quotient algebra  $C < t, D^{-1} > / R_{tt} \cup R_{tD^{-1}}$  is a Hopf algebra with the co-product  $\Delta$ , the co-unit  $\epsilon$  and antipode  $S$  defined by :

$$\Delta(T) \equiv T \otimes T, \quad \Delta(D^{-1}) \equiv D^{-1} \otimes D^{-1} \quad (25)$$

$$\epsilon(T, D^{-1}) \equiv (I, 1), \quad S(T) \equiv T^{-1}, \quad S(D) \equiv D^{-1} \quad (26)$$

When  $x^1$  and  $x^2$  are mutually adjoint and when  $x^3$  is self-adjoint (for instance, in the case of the  $q$ -oscillator algebra), the relations  $R_{xx}$  are unchanged if the parameters are real. The action of  $T$  respects this property if the quantum group is equipped with a star-operation that is an antihomomorphism such as

$$\begin{aligned} ((t_j^i)^*)^* &= t_j^i, & \forall i, j \\ (t_2^2)^* &= t_1^1, & (t_1^2)^* &= t_2^1, & (t_3^2)^* &= t_3^1, \\ (t_1^3)^* &= t_2^3, & (t_2^3)^* &= t_3^3. \end{aligned} \quad (27)$$

These relations are consistent with (22) and (24) and the quantum group  $C < t, D^{-1} > / R_{tt} \cup R_{tD^{-1}}$  acquires the structure of a Hopf-star-algebra.

Let us stress some properties of  $C < t, D^{-1} > / R_{tt} \cup R_{tD^{-1}}$  :

- When  $s = 0$ , the variable quantum space is the three-dimensional quantum plane, the resulting quantum group corresponds to an original deformation of  $GL(3)$  [7], [6].

- When  $t_1^3$  and  $t_2^3$  vanish, we obtain a deformation  $G_{qs}$  of the subgroup  $G$  of  $GL(3)$ . Two of the relations  $R_{tt}$  give :

$$(u^2 - q) t_2^1 t_3^3 = 0,$$



and

$$(u^2 - q) t_1^2 t_3^3 = 0.$$

implying that  $q$  is equal to  $u^2$  if the algebra has no zero divisors. When  $q = u^2$ , the matrix  $\Omega$  being equal to its inverse, the two differential calculi on  $C < x > / R_{xx}$  reduce to one. This differential calculus was previously obtained by a completely different method, implying the uniqueness of the result [11]. All the commutation relations are invariant under the ten-generator quantum group  $C < t, D^{-1} > / R_{tt} \cup R_{tD^{-1}}$  and under a quantum group  $G_{qs}$  deduced from the previous one by putting  $t_1^3 = t_2^3 = 0$ .

- The relations (27) are consistent with the condition  $t_1^3 = t_2^3 = 0$ , and  $G_{qs}$  is a Hopf-star-subalgebra of  $C < t, D^{-1} > / R_{tt} \cup R_{tD^{-1}}$ .

- In the case where  $q = u^2$ , we would point out that, if we add the generator  $(t_3^3)^{-1}$  to the quantum group  $G_{qs}$ , the  $T$ -matrix can be written on the form  $T' \times t_3^3$  with  $t_j^i = t_j^i (t_3^3)^{-1}$ . All the elements  $t_j^i$  commute two by two. Nevertheless, they cannot be identified to C-numbers (and then  $T'$  to a matrix belonging to the initial subgroup  $G$ ), because if they were C-numbers all the elements  $t_j^i$  would be proportional to  $t_3^3$  and this is impossible due to their commutation relations resulting from (22).

## 4 Conclusion

Two differential calculi can be associated with the three-parameter oscillator algebra  $C < x > / R_{xx}$  and in particular with the q-oscillator algebra. They are invariant under the same quantum group  $C < t, D^{-1} > / R_{tt} \cup R_{tD^{-1}}$  that is an original three-parameter deformation of  $GL(3)$ . When we assume that the variables  $x^1$  and  $x^2$  are mutually adjoint and that  $x^3$  is self adjoint, a star-operation is defined on the invariance quantum group that then becomes a Hopf-star-algebra. Finally we consider the case where two generators,  $t_1^3$  and  $t_2^3$ , are removed from the algebra  $C < t, D^{-1} > / R_{tt} \cup R_{tD^{-1}}$  and we prove that the resulting differential calculus and quantum group of invariance do not exist for arbitrary values of the parameters, this is in agreement with a previous work [10][11].

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